Extending Portfolio Theory to Compound Returns

Hendrik Bessembinder*

W.P. Carey School of Business

Arizona State University

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Abstract

Solutions are presented for the mean, variance, and skewness of compound portfolio returns, with and without periodic rebalancing, in a setting where single-period returns are symmetric. More frequent rebalancing reduces portfolio volatility and is unambiguously preferred by mean-variance investors in the absence of investment skill. However, more frequent rebalancing and broader diversification both reduce compound return skewness. An investor with a sufficiently strong taste for skewness may prefer a more concentrated portfolio that is rebalanced less frequently, even in the absence of investment skill, particularly if the investment horizon is long.

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1. Introduction

Modern portfolio theory (MPT), originated by Markowitz (1952) and presented in most standard finance texts, provides important insights into how investment risk and return are affected by the diversification of portfolio investment across securities. Most applications of MPT focus on the mean and volatility (standard deviation or variance) of single-period portfolio returns.[1] Further, the parameters of single period return distributions are most often estimated over a relatively short horizon, such as monthly.

Of course, many investors care about long-horizon portfolio outcomes. In this paper, I consider portfolio returns that are compounded over multiple periods. I also extent the analysis beyond mean and volatility to consider the skewness of portfolio returns.[2] While the literature has previously considered the potential role of investors' preferences for skewness in returns, the degree of skewness in the short-horizon returns that are typically studied tends to be relatively modest.[3] In contrast, the skewness of compound long-horizon returns can be strikingly large even if skewness is minimal or entirely absent in single-period returns, as Bessembinder (2018) and Farago and Hjalmarsson (2021) show. To my knowledge this paper is the first to develop expressions to quantify the interplay between diversification, rebalancing frequencies, volatility, and skewness in long-horizon portfolio returns.

It is an intriguing question why the extension of MPT to compound returns has not received more attention from researchers in the nearly seventy years since the appearance of Markowitz' seminal work. Misinterpretations of the analysis provided by Samuelson (1969) may contribute to the explanation. Samuelson showed that a multiperiod investor rationally behaves *myopically*.[4] Specifically, in

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[1] MPT has been criticized for failing to allow for "fat tails" in short-horizon returns, the presence of which has been documented at least since Mandelbrot (1963). This oversight can only be strictly justified by the tenuous assumption that investor preferences can be described by quadratic utility functions.

[2] Gunthorpe and Levy (1994) foreshadow the desirability of considering the skewness of long-horizon returns when they show that mean-variance analysis implies that investors with longer time horizons should invest a *smaller* proportion of their wealth in risky assets. The authors caution against applying mean-variance analysis to long investment horizons, but do not go on to explicitly consider skewness.

[3] See, for example, Harvey, C., J. Liechty, M. Liechty, and P. Muller (2010), Harvey and Siddique (2000), Krauss and Litzenberger (1976) Patton (2004) and the papers referenced there.

[4] Samuelson (1969) did not use the term "myopic," but it has been adopted by the literature (e.g. Campbell and Viceira, 2002) to describe investor strategies under Samuelson's assumptions.

Samuelson's model the investor selects portfolio weights that are optimal in light of single-period return parameters, and then rebalances the portfolio each period so as to return to the same weights. Stated alternatively, Samuelson showed that knowledge regarding compound multiperiod return parameters contributes nothing useful to an investor who already knows the parameters of single-period returns.

There are, however, important limitations to Samuelson's analysis that may not be fully appreciated. First, his outcomes require that the distribution of investment returns is independent and identical (iid) over time. Second, his prescription applies to investors with one particular objective, the maximization of the expectation of a "power" (also referred to as isoelastic or constant relative risk aversion) utility function. While a substantive literature has arisen to study multi-period investing, most follow Samuelson's power utility assumption.[5] That is, it could be argued that the literature mainly studies multi-period investing under the specific assumption that minimizes the importance of the issues that arise due to compounding. At the risk of stating the obvious, objectives can differ across disparate investors and investment managers. Investors may maximize the expectation of a traditional convex utility function that differs from the power specification. Or, investor behavior might be better described by non-traditional preferences, such as those embodied by prospect theory, originally attributable to Kahneman and Tversky (1979) and recently applied to empirical asset pricing anomalies by Barberis, Jin and Wang (2021). Further, investment decisions are often delegated to professional managers, whose portfolio selections may depend in part on their compensation structure.

The third limitation is distinct from the fact that Samuelson's analysis rests on knife-edge assumptions, and is arguably the most important. If some investors follow Samuelson's rebalancing prescription by selling assets that have outperformed and purchasing those that have underperformed so as to return to initial weights, then other investors necessarily trade the opposite direction. That is, it *cannot* be the case that Samuelson's myopia prescription applies to all investors, or to the market as a whole. Stated alternatively, the assertion that a rational multi-period investor is myopic can only apply, at

[5] See, for example, Campbell and Viceira (2002), Barberis (2000), and Ang (2014). Thorley (1995) summarizes some specific criticisms of power utility applied to multi-period investing.

best, to a non-representative segment of the investor population. The parameters of compound long-horizon returns will generally be relevant to investors with other objectives, to investors who take the opposite side of myopic investors' rebalancing trades, and to the market as a whole, even if returns are iid.

I present in the Mathematical Appendix to this paper equations to compute the mean, variance, and skewness of portfolio returns for various investment horizons, and for alternative rebalancing frequencies within the investment horizon. In the paper most similar to this one in terms of its focus on compound portfolio returns, Farago and Hjalmarsson (2021b) use simulations to obtain predictions, and also provide empirical evidence, regarding the likelihood that compound returns to periodically-rebalanced portfolios will exceed those to buy-and-hold portfolios. However, to my knowledge the prior literature has not specified the precise relations between portfolio diversification, rebalancing, and skewness in compound returns, as I do here.[6] The results presented here quantify the strong positive skewness in compound long-horizon returns that arise even when single-period returns are symmetric, and also show the extent to which periodic rebalancing reduces the degree of such skewness.

To obtain exact solutions, I rely on simplifying assumptions, including that the statistical distribution of individual stock single-period returns is independent and identical (iid) over time, that individual stock returns each period are determined by a single-factor market model, that single-period returns are symmetric (zero skew) and that rebalancing, if it occurs, is to the same portfolio weights that were selected initially. Despite these simplifying assumptions, the expressions in the Appendix allow for considerable flexibility, including that single-period alphas, betas, and idiosyncratic volatilities can differ across stocks.

For illustrative purposes I impose additional simplifications. In particular, the portfolio initially places equal weight on each stock, the alpha for every stock is zero, the beta for every stock is one, the

[6] Farago and Hjalmarsson (2021a) derive expressions for the skewness of compound returns to assets when single-period returns that are iid. However, buy-and-hold returns are generally not iid (even if component returns are iid), because portfolio weights vary randomly through time, depending on realized returns.

mean single-period market return (and given the preceding assumptions, each stock's mean single-period return) is 0.75%, the standard deviation of market returns is 8%, and the standard deviation of each stock's firm specific return is 10%. These parameters are selected so that the single-period might reasonably be thought of as comprising one month. Under these assumptions, in particular that returns are iid and single-period alphas are zero, there is no scope for portfolio manager skill in terms of selecting stocks with better long run potential or shunning stocks with worse long run potential. Despite this limitation, the results are of interest in terms assessing the pure effects of compounding and rebalancing on portfolio outcomes.

I illustrate outcomes for investment horizons ranging from twelve to 240 months (one to 20 years) and for portfolios containing between five and 300 stocks. For each, I consider outcomes that are attained if portfolios are initially equal-weighted and are not rebalanced at any point within the investment horizon (i.e. buy-and-hold outcomes), as well as outcomes that are attained if the portfolio is periodically rebalanced by selling those stocks that have performed better and purchasing stocks that have performed worse to re-attain equal weights. I consider rebalancing intervals that allow for a whole number of such intervals within the investment horizon. For example, when the investment horizon is 240 months, I consider rebalancing intervals of 1 month, 12 months, 24 months, 48 months, 60 months, and 120 months.

Given these assumptions, the mean portfolio return at each horizon is independent of both the number of securities and the rebalancing frequency, and is simply the mean monthly return (which is common to all securities and portfolios) compounded for the indicated number of periods. With a mean monthly return of 0.75%, the mean portfolio return is 9.38% at the annual horizon, 56.57% at the 5-year horizon, 145.14% at the 10-year horizon, and 500.92% at the twenty-year horizon.

2. Investment Horizon, Diversification, Return Volatility, and Skewness.

a. Buy-and-Hold Portfolios

Table 1A displays standard deviations of buy-and-hold returns for portfolios containing between five and 300 stocks, for investment horizons ranging from one to 240 months, and Figure 1A displays the same data graphically. Volatility increases with investment horizon. It can be verified that the data in Table 1A implies that the growth in the variance of portfolio returns (the square of the standard deviation) is *more* than proportional to the horizon.[7] This fact is at odds with some interpretations of the notion of "time diversification."[8]

As is well known from MPT, volatility, as measured by portfolio return standard deviation, decreases with the number of stocks in the portfolio.[9] One insight gained from examining the data in Table 1A is that the effect of diversification is more pronounced at long horizons. Focusing, for example, on outcomes when the number of stocks is increased from five to 25, the reduction in the portfolio standard deviation at the monthly horizon is 10.0% (from 9.2% to 8.2%), while the reduction in the portfolio standard deviation at the twenty-year horizon is 34.0% (from 2080.1% to 1371.9%). Thus, simple rules of thumb developed by considering short-horizon returns, along the lines of "ninety percent of the benefit from diversification is obtained with only 30 stocks" do not cleanly carry over to long-horizon returns.[10]

Table 2A displays portfolio buy-and-hold return skewness coefficients for portfolios containing between five and 300 stocks, for investment horizons ranging from one to 240 months, while Figure 2A displays the same data graphically.[11] The data shows that the skewness of portfolio returns increases dramatically with investment horizon, even though single-period returns have zero skewness, and more so for less diversified portfolios. For example, the skewness of returns to 100-stock portfolios increases

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[7] In contrast, the variance of continuously compounded (logarithmic) returns grows proportionate to the investment horizon if returns are iid. However, since the actual portfolio return is obtained as the non-linear exponential transformation of the sum of the log returns, this well-known proportionality result does not carry over to actual returns.

[8] Samuelson (1969) attempted to put the notion of time diversification to rest when (noting that long-horizon gross returns are obtained by multiplying successive gross short-horizon returns) he referred to "the mistaken notion that multiplying the same kind of risk leads to cancellation rather than augmentation of risk." Of course, if returns display negative serial dependence (i.e., prices are mean reverting) then long-horizon return volatility is dampened as compared to the iid benchmark evaluated here.

[9] The reduction in risk as the number of stocks is increased as displayed on Figure 1A may seem less dramatic as compared to Figures often displayed in texts. This reflects that the graph displays standard deviations rather than variances, and excludes observations for portfolios with less than five stocks.

[10] Among references to statements of this type the earliest paper appears to be Fisher and Lorie (1970).

[11] To be specific, the return skewness coefficient displayed is the third central moment divided by the cube of the standard deviation.

from zero at the monthly horizon to 0.80 at the annual horizon, 4.75 at the decade horizon, and 23.09 at the 20-year horizon, while the skewness of returns to 10-stock portfolios grows much more rapidly, from zero at the monthly horizon to 0.90 at the annual horizon, 7.93 at the decade horizon, and 82.89 at the 20-year horizon.

MPT is predicated on the idea that investors prefer a higher mean return and lower return volatility. However, traditional economic theory focused on rational utility maximization allows that investors may prefer positive skewness in the distribution of portfolio returns.[12] Further, non-traditional objectives such as those described by prospect theory imply that investors will prefer assets with positively skewed returns (e.g., Barberis, Jin and Wang, 2021). A considerable body of empirical evidence supports the notion that investors prefer assets with positively skewed distributions. The data in Table 2A allows that an investor with a sufficiently strong preference for skewness could desire a narrow rather than a broadly diversified portfolio, particularly if their investment horizon was long rather than short.[13]

b. Monthly Rebalanced Portfolios

The potential benefits of periodic portfolio rebalancing have been discussed at least since Booth and Fama (1992). As Ang (2014) notes, rebalanced portfolios will tend to outperform buy-and-hold portfolios if returns display systematic reversion over time, and will underperform if returns display systematic continuation or trends. Here, I focus on iid returns, where rebalanced portfolios do not display systematically higher or lower mean returns as compared to buy-and-hold portfolios.

Tables 1B and 2B report results that correspond to those in Tables 1A and 2A, except that they pertain to portfolios that are rebalanced to equal weights every month. Figures 1B and 2B display the same data graphically. Comparing results across Tables 1A and 1B, it can be seen that monthly

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[12] See, for example, Brunnermeier, Gollier and Parker (2007), Conrad, Dittmar and Ghysels (2013), Harvey and Siddique (2000), Krauss and Litzenberger (1976), Mitton and Vorkink (2007), and Patton (2004).

[13] The only prior study of which I am aware that considers the tradeoff between return volatility and skewness as a function of diversification across assets is that provided by Xiong and Idzorek (2019), who focus on short horizon, not compound long-horizon, returns.

rebalancing moderates the effect of return horizon on portfolio risk. That is, while the standard deviation of portfolio returns still increases with investment horizon, the rate of increase is slower than for buy-and-hold portfolios. For example, for a portfolio with ten stocks, the standard deviation of monthly-rebalanced portfolio returns (Table 1B) grows from 8.6% at the monthly horizon to 33.0% at the annual horizon, 289.1% at the decade horizon, and 1305.0% at the twenty-year horizon, while the standard deviation of buy-and-hold portfolio returns comprised of ten stocks (Table 1A) is 8.6% at the monthly horizon, 33.1% at the annual horizon, 309.9% at the decade horizon, and 1673.0% at the twenty-year horizon.

Rebalancing reduces long-horizon portfolio volatility because it effectively restores portfolio diversification. The strong positive skewness in the distribution of individual stock returns at long horizons implies there will be a few stocks with very large long-horizon returns, even while most stocks have moderate returns, which in turn implies that a buy-and-hold portfolio effectively becomes less diversified over time as the weights on winning stocks increase.

Farago and Hjalmarsson (2021a) show that the skewness of long-horizon returns depends primarily on the volatility of short-horizon returns. Since a comparison of the results in Tables 1A and 1B shows that monthly rebalancing reduces the rate at which portfolio return volatility increases with investment horizon, it may be anticipated that monthly rebalancing also reduces the rate at which portfolio return skewness increases with investment horizon. Comparing results across Tables 2A and 2B verifies this intuition. Focusing again on portfolios containing ten stocks, the skewness of rebalanced portfolio returns increases from zero at the monthly horizon to 0.84 at the annual horizon, 5.02 at the decade horizon, and 16.09 at the twenty-year horizon, while the corresponding skewness figures for the buy-and-hold portfolio are zero at the monthly horizon, 0.90 at the annual horizon, 7.93 at the decade horizon, and 82.89 at the twenty-year horizon.

These results verify that the desirability of periodic portfolio rebalancing depend on investor preferences.[14] Periodic rebalancing reduces portfolio return volatility and portfolio return skewness, with the strongest effects apparent with longer investment horizons. An investor who is averse to volatility and has no particular interest in skewness would (given the other assumptions employed, including zero alphas and iid returns) unambiguously prefer that portfolios be rebalanced regularly. On the other hand, an investor with a sufficiently strong skewness preference could prefer that portfolios not be rebalanced. These considerations are most important for investors with long investment horizons.

3. Intermediate Rebalancing Intervals and Sharpe Ratios

The results reported in Section 2 pertain to buy-and-hold portfolios and to portfolios that are rebalanced every month. I next assess the effects of intermediate rebalancing intervals. Table 3 reports outcomes for an investment horizon of 20 years, for rebalancing intervals of one month, 12 months, 24 months, 48 months, 60 months, 120 months, and 240 months, the last of which is buy-and-hold for the full 20 years. Figures 3A to 3D display the same data graphically. Table 4 and Figures 4A to 4D display corresponding results for a 10-year investment horizon, and generally support the same conclusions.

Considering first the portfolio return standard deviation, it can be observed on Panel A of Table 3 that more frequent rebalancing always reduces portfolio volatility, and more dramatically so for portfolios with fewer stocks. For example, rebalancing every year rather than every ten years reduces the 20-year portfolio return standard deviation of a ten-stock portfolio by 8.8% (from 1,441% to 1,314%), while rebalancing every year rather than every ten years reduces the volatility of a 100-stock portfolio by only 1.2% (from 1,159% to 1,145%).

Sharpe (1966) proposed the ratio of the mean portfolio return to the standard deviation of portfolio return as a performance measure that embodies both the average return as well as return volatility. However, the portfolio with the highest "Sharpe ratio" does not necessarily maximize an

[14] Of course, the specific results here depend in part on the iid assumption. Systematic reversals (continuations) in returns at the relevant horizons would increase (decrease) the desirability of rebalancing.

investor's expected utility, except under restrictive assumptions (e.g. Zakamouline and Koekebakker, 2009). Still, Sharpe ratios are broadly employed in practice.

Sharpe ratios are difficult to interpret when computed over differing investment horizons, because increasing the horizon affects the mean return and the standard deviation of returns differentially. However, Sharpe ratios pertaining to a given investment horizon can be compared across different numbers of stocks and different rebalancing intervals. Sharpe ratios for the 20-year horizon are displayed on Panel B of Table 3. As would be expected in the current setting that does not allow for possible manager skill, Sharpe ratios always decrease as the portfolio is diversified by adding more stocks. Further, Sharpe ratios always increase with more frequent rebalancing, as frequent rebalancing restores effective diversification, as previously discussed. For example, the Sharpe ratio of a ten-stock portfolio with a 20-year investment horizon is 0.30 without rebalancing, 0.35 with rebalancing after ten years, and 0.38 with annual rebalancing.

Sharpe ratios do not consider portfolio return skewness, which can also be relevant to investors. The data reported on Panel C of Table 3, which pertains to the 20-year investment horizon, verifies that more frequent rebalancing always reduces portfolio return skewness. The effect is more dramatic for portfolios with fewer stocks, to an even greater extent than for return volatility. For example, rebalancing every year rather than every ten years reduces the 20-year portfolio return skewness of a ten-stock portfolio by essentially half (from 33.16 to 16.75), while rebalancing every year rather than every ten years reduces the skewness of a 100-stock portfolio by 11.0% (from 13.79 to 12.27).

The Sharpe ratio can be extended to consider the role of skewness preference, as in the "Adjusted-for-Skewness Sharpe Ratio (ASSR)" proposed by Zakamouline and Koekebakker (2009). If SR denotes the Sharpe Ratio (mean return divided by standard deviation of return) and SKEW denotes the standardized skewness coefficient (third central moment divided by standard deviation cubed), then ASSR = SR[1 + b*SR*SKEW/3]. Here, b is a parameter that describes an individual's degree of skewness preference. I illustrate how diversification, compounding and rebalancing can affect the ASSR while assuming b = 1.[15]

Panel D of Table 3 reports Adjusted-for-Skewness Sharpe Ratios based on the parameters described earlier. Two points are noteworthy. First, ASSRs increase as portfolios are rebalanced *less* frequently, a result that reflects the rapid increase in portfolio return skewness over longer horizons in non-rebalanced portfolios. With N=50 stocks, for example, the ASSR is 1.21 with monthly rebalancing, 1.35 with rebalancing after ten years, and 2.14 when the portfolio is not rebalanced within the 20-year investment horizon. Second, while Sharpe Ratios always increase as the portfolio is diversified across more stocks, and the same result is observed for ASSRs over most parameters considered, there are exceptions. For example, with monthly rebalancing, the ASSR increases from 1.15 with N = 5 stocks to 1.20 with N = 30 stocks, while, in contrast, with rebalancing every 60 months the ASSR *decreases* from 1.41 with N = 5 stocks to 1.26 with 30 stocks. The highest ASSR observed for the range of parameters considered in Panel D of Table 3 is 2.78, for a narrow portfolio of N = 10 stocks, and without rebalancing. Since relations between portfolio breadth, rebalancing frequencies, and the ASSR are not monotone, the ASSR can, for at least some parameters, be maximized at intermediate levels of diversification and rebalancing. More broadly, this analysis supports the reasoning that investors with a sufficiently strong skewness preference may rationally prefer concentrated portfolios that are rebalanced less frequently to more diversified portfolios or portfolios that are rebalanced more frequently.

However, some limitations of the illustrations presented here should be noted. The ASSR does not consider kurtosis or higher-order moments that may also be relevant to investors. It is therefore similar in principal to approaches such as Mitton and Vorkink (2007) that focus on third-order approximations to more general utility functions, and that view a positive third derivative of the utility function with respect to wealth as indicating a preference for skewness. A third-order approximation by construction considers the mean, variance, and skewness of outcomes, but not higher-order moments.

[15] Zakamouline and Koekebakker (2009) note that b = 1 is consistent with a utility function that displays constant absolute risk aversion.

Farago and Hjalmarsson (2021a) show that portfolio positions selected based on such a third-order approximation can fail to maximize expected utility, and attribute this outcome in part to the failure to consider the fact that longer compounding periods alter kurtosis and higher moments even as they increase return skewness. Further, they show (in line with the findings of Samuelson, 1969) that, despite the fact that the third derivative of a power utility function is positive, under simple assumptions similar to those employed here, investors with power utility functions will not display any skewness preference. Stated alternatively, with iid single-period returns the skewness observed in compound returns would be attractive to investors with *stronger* skewness preference as compared to that implied by power utility. Of course, as noted, investors tastes may reflect non-standard objective functions. Further, a preference for skewness could arise from extrinsic motivations, such as a non-linear relation between performance and compensation or fund flows.

4. Conclusions

Modern Portfolio Theory (MPT) emphasizes the desirability of diversification, particularly for investors who lack skill to identify stocks with positive or negative alpha. However, MPT has mainly been focused on returns measured over a single period, and in most applications the period is relatively short, e.g. one month or one year. Of course, some investors care about the returns that accrue over much longer horizons.

The skewness of short horizon returns is relatively modest, which can help to justify an exclusive focus on the mean and volatility of short-horizon portfolio returns. However, recent research, including Bessembinder (2018) and Farago and Hjalmarsson (2021) has emphasized that long-horizon stock returns are characterized by large positive skewness, even if short horizons returns display little or no skewness. Economic theory allows that investors may prefer positive skewness, which is abundant over longer horizons.

The results obtained in this study show that a long-horizon investor who has no skill in identifying stocks with non-zero alpha and who cares only about the mean and volatility of portfolio returns will prefer both to include a large number of stocks in their portfolio to achieve broad diversification and to rebalance the portfolio periodically so that the skewness of stock returns over longer horizons does not effectively diminish the volatility-reducing benefits of diversification. However, an investor with a sufficiently strong preference for skewness could view the situation differently. The inclusion of a large number of stocks in the portfolio and periodic rebalancing both reduce the skewness of long investment horizon returns, as well as return volatility. If the investor's preference for skewness is sufficiently strong as compared to their aversion to volatility, they could rationally (depending on their attitudes toward other parameters such as kurtosis) prefer a narrower portfolio with less frequent rebalancing. More broadly, both diversification into more stocks and periodic rebalancing involve a tradeoff between skewness and volatility that is particularly pronounced for long-horizon investors.

The results presented here are rely on assumptions, including iid returns over time and that each stock's single-period alpha is zero, that leave no room for investment skill. It will be of particular interest to assess how the results presented here are altered by allowance for varying degrees of investor skill.

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Mathematical Appendix

Part I: The Mean, Variance, and Skewness of Compound returns to a Buy-and-Hold Portfolio

Consider a portfolio containing N stocks. Let **W** denote the Nx1 vector of portfolio weights, $\boldsymbol{\mu}^{\mathbf{L}}$ denote the Nx1 vector of portfolio mean returns with typical element μ_i^L, $\mathbf{V}^{\mathbf{L}}$ denote the NxN return covariance matrix with typical element $COV(R_i^L, R_j^L) = E[(R_i^L - \mu_i^L)(R_j^L - \mu_j^L)]$, where E denotes expectation, R denotes return, the subscript refers to a particular stock, and the superscript L is used to denote that the return is measured over a long horizon. Also, adopting the terminology of Martellini and Ziemann (2010), let $\mathbf{S}^{\mathbf{L}}$ denote the NxN^2 "tensor" matrix with typical element:

$$s_{ijk} = E[(R_i^L - \mu_i^L)(R_j^L - \mu_j^L)(R_k^L - \mu_k^L)].$$

The fact that returns are measured over a long rather than a short horizon does not, itself, alter the basic mathematics of Modern Portfolio Theory, which can be applied to state the mean, variance, and skewness, respectively, of the portfolio return as:

$$E(R_p^L) = W'\mu^L \tag{1}$$

$$VAR(R_p^L) = W'V^L W \tag{2}$$

$$SKEW(R_p^L) = W'S^L(W \otimes W'). \tag{3}$$[16]

The specific challenge here is to identify the elements of the matrices $\boldsymbol{\mu}^{\mathbf{L}}$, $\mathbf{V}^{\mathbf{L}}$ and $\mathbf{S}^{\mathbf{L}}$ as functions of parameters that apply to the individual shorter periods that comprise the long horizon. I employ the following assumptions regarding single-period returns. The gross (one plus) return to each stock is given by the single factor market model:

$$R_i = \alpha_i + \beta_i M + \varepsilon_i,$$

where α_i is one plus the traditional alpha, β_i is the stock's beta, or sensitivity to M, a common market factor with mean μ_M and variance σ_M^2, and ε_i is a zero-mean error with variance σ_i^2. This specification implicitly assumes a zero risk-free rate of interest. I assume the errors are mutually independent across stocks and independent of the market, and that both the errors and the market factor are distributed symmetrically, i.e. have zero skew, in each single-period.

Then, the expected gross stock *i* return is

$$E(R_i) = \alpha_i + \beta_i \mu_M \tag{4}$$

and the covariance of single-period returns across pairs of stocks is

$$COV(R_j, R_k) = \beta_j \beta_k \sigma_M^2 + COV(\varepsilon_j, \varepsilon_k) \tag{5}$$

Where $COV(\varepsilon_j, \varepsilon_k) = 0$ for all $j \neq k$.

[16] To be more precise, expression (3) provides the third central moment of portfolio returns. The skewness coefficient is typically defined by standardizing the third central moment, in particular dividing by the variance to the 3/2 power.

Using the definitions of expectations and covariances, the following can also be obtained.

$$E(R_j R_k) = E(R_j)E(R_k) + COV(R_j, R_k) \tag{6}$$

$$COV(R_i, R_j R_k) = E(R_k)COV(R_i R_j) + E(R_j)COV(R_i R_k). \tag{7}$$

I also assume that returns for each security and the market are independently and identically distributed over time. Compound (long-horizon) returns are products successive single-period gross returns.

Let the L superscript specifically denote returns compounded over L periods. The mean long-horizon return for a typical stock is

$$\mu_i^L + 1 \equiv E(R_i^L) = [E(R_i)]^L. \tag{8}$$

Consider two variables, a and b, that may have non-zero contemporary covariance with each other, but are each distributed iid over time. Let the subscript t denote time t observations. We are interested in the covariance of their products over L observations:

$$COV_{a,b}^L = COV(a_1 a_2 a_3 \ldots\ldots a_L, b_1 b_2 b_3 \ldots\ldots b_L).$$

Using the definition of covariance this is

$$COV_{a,b}^L = E(a_1 b_1 a_2 b_2 a_3 b_3 \ldots\ldots a_L b_L) - E(a_1 a_2 a_3 \ldots\ldots a_L)E(b_1 b_2 b_3 \ldots\ldots b_L).$$

Using independence over time this can be stated as:

$$COV_{a,b}^L = [COV(a,b) + E(a)E(b)]^L - E(a)^L E(b)^L. \tag{9}$$

The application of (9) when a = R_i and b = R_j gives (as shown by Levhari and Levi, 1977) the typical element of the $\mathbf{V^L}$ matrix as:

$$COV(R_i^L, R_j^L) = [COV(R_i, R_j) + E(R_i)E(R_j)]^L - [E(R_i)E(R_j)]^L. \tag{10}$$

The application of (9) when a = R_i and b = $R_j R_k$ gives

$$COV(R_i^L, R_j^L R_k^L) = [COV(R_i, R_j R_k) + E(R_i)E(R_j R_k)]^L - [E(R_i)E(R_j R_k)]^L. \tag{11}$$

Also, using the definitions of expectation and covariance, it can be shown that the typical element of the tensor matrix defined above is:

$$S_{ijk} = COV(R_i^L, R_j^L R_k^L) - E(R_k^L)COV(R_i^L, R_j^L) - E(R_j^L)COV(R_i^L, R_k^L). \tag{12}$$

Expressions (8), (10), and (11) can be substituted into (12) to define the elements of the tensor matrix $\mathbf{S^L}$, at which point expressions (1), (2), and (3) can be employed to obtain the mean, variance, and skewness of the compound buy-and-hold portfolio returns.

Part II: The Mean, Variance, and Skewness of Compound returns to a Rebalanced Portfolio

Consider an asset with a gross return, R_p, mean return μ_p, variance σ_p^2, and skewness γ. Assume that R_p is distributed independently and identically (iid) over time. The variance of R_p compounded over T periods can be obtained by applying (10):

$$VAR(R_p^T) = [\sigma_p^2 + \mu_p^2]^T - [\mu_p^2]^T. \tag{13}$$

Farago and Hjalmarsson (2021a) show (their expression 7) that, with the iid assumption, the skewness of R_p compounded over T periods is

$$SKEW\left(R_p^T\right) = \frac{2+\beta_3^T-3\beta_2^T}{(\beta_2^T-1)^{3/2}} \tag{14}$$

Where $\beta_2 = 1 + \frac{\sigma_p^2}{\mu_p^2}$ and $\beta_3 = -2 + 3\beta_2 + \gamma(\beta_2 - 1)^{3/2}$.

Assuming that individual security returns are iid, the returns to a portfolio that is rebalanced to equal weights each period are also iid, so expressions (13) and (14) can be applied to obtain the variance and skewness of compound returns to rebalanced portfolios. In contrast, the returns to a buy-and-hold portfolio are not generally iid across periods because portfolio weights evolve randomly as a function of realized period-by-period returns, so expressions (13) and (14) for rebalanced portfolios are not equivalent to expressions (2) and (3) for buy-and-hold portfolios.

Let M denote the total investment horizon. Let L denote the number of periods that pass before the portfolio is rebalanced to its initial weights. If L = M then the portfolio is not rebalanced within the horizon of interest, so expressions (1) to (3) apply. If L = M/T then the portfolio is rebalanced T times. For example, if M = 240 months and T = 4 then the portfolio is rebalanced every L = 60 months. In this case expressions (1) to (3) can be implemented with L = 60 to obtain the portfolio mean, variance, and skewness for each L = 60-month interval during which there is no rebalancing. With these in hand, expressions (13) and (14) can be implemented for the T = 4 consecutive 60-month intervals. The results provide parameters for portfolio outcomes over M=240 periods with rebalancing every L = 60 months.

Table 1A: Buy and Hold Portfolio Return Standard Deviation by Investment Horizon and Number of Stocks

Number Stocks	Investment Horizon (Months)																				
	1	12	24	36	48	60	72	84	96	108	120	132	144	156	168	180	192	204	216	228	240
5	9.2%	35.5%	56.6%	78.4%	102.6%	130.0%	161.7%	198.7%	241.9%	292.8%	352.6%	423.4%	507.0%	606.1%	723.7%	863.4%	1029.5%	1227.3%	1463.1%	1744.3%	2080.1%
10	8.6%	33.1%	52.6%	72.5%	94.2%	118.6%	146.5%	178.7%	216.0%	259.3%	309.9%	368.9%	438.0%	519.0%	614.2%	726.2%	858.1%	1013.8%	1197.7%	1415.3%	1673.0%
15	8.4%	32.3%	51.2%	70.4%	91.2%	114.5%	141.1%	171.5%	206.6%	247.2%	294.2%	348.8%	412.4%	486.6%	573.1%	674.3%	792.8%	931.8%	1095.0%	1287.0%	1513.2%
20	8.3%	31.9%	50.5%	69.3%	89.7%	112.5%	138.3%	167.8%	201.8%	240.9%	286.1%	338.4%	399.0%	469.5%	551.4%	646.8%	758.0%	888.0%	1039.9%	1217.8%	1426.5%
25	8.2%	31.6%	50.1%	68.6%	88.7%	111.2%	136.6%	165.6%	198.8%	237.0%	281.1%	331.9%	390.8%	458.9%	538.0%	629.7%	736.4%	860.6%	1005.4%	1174.4%	1371.9%
30	8.2%	31.5%	49.8%	68.2%	88.1%	110.3%	135.5%	164.1%	196.8%	234.4%	277.7%	327.6%	385.2%	451.8%	528.8%	618.1%	721.6%	841.8%	981.7%	1144.5%	1334.3%
35	8.2%	31.4%	49.6%	67.9%	87.7%	109.7%	134.6%	163.0%	195.4%	232.5%	275.2%	324.4%	381.1%	446.6%	522.2%	609.6%	710.8%	828.2%	964.4%	1122.6%	1306.7%
40	8.2%	31.3%	49.4%	67.6%	87.3%	109.3%	134.0%	162.1%	194.3%	231.1%	273.4%	322.0%	378.1%	442.6%	517.1%	603.2%	702.7%	817.8%	951.2%	1106.0%	1285.7%
50	8.1%	31.1%	49.2%	67.3%	86.9%	108.6%	133.1%	161.0%	192.7%	229.1%	270.8%	318.7%	373.7%	437.1%	510.0%	594.1%	691.1%	803.1%	932.5%	1082.2%	1255.6%
75	8.1%	31.0%	48.9%	66.8%	86.2%	107.7%	131.9%	159.4%	190.7%	226.4%	267.3%	314.1%	367.9%	429.5%	500.3%	581.7%	675.3%	782.9%	906.9%	1049.7%	1214.4%
100	8.1%	30.9%	48.7%	66.6%	85.9%	107.3%	131.4%	158.6%	189.6%	225.0%	265.5%	311.8%	364.9%	425.7%	495.4%	575.4%	667.3%	772.7%	893.8%	1033.1%	1193.2%
200	8.0%	30.7%	48.5%	66.3%	85.4%	106.6%	130.5%	157.4%	188.1%	223.0%	262.8%	308.4%	360.4%	419.9%	488.0%	565.9%	655.0%	757.0%	873.8%	1007.6%	1160.8%
300	8.0%	30.7%	48.4%	66.2%	85.3%	106.4%	130.2%	157.0%	187.5%	222.3%	261.9%	307.2%	358.9%	418.0%	485.5%	562.7%	650.9%	751.8%	867.1%	998.9%	1149.8%

Table 1B: Portfolio Return Standard Deviation with Monthly Rebalancing, by Investment Horizon and Number of Stocks

Number Stocks	Investment Horizon (Months)																				
	1	12	24	36	48	60	72	84	96	108	120	132	144	156	168	180	192	204	216	228	240
5	9.2%	35.3%	56.0%	76.9%	99.7%	125.2%	154.1%	187.2%	225.0%	268.5%	318.5%	376.1%	442.5%	519.0%	607.0%	708.5%	825.5%	960.2%	1115.3%	1293.9%	1499.6%
10	8.6%	33.0%	52.2%	71.5%	92.5%	115.7%	142.0%	171.8%	205.7%	244.6%	289.1%	340.1%	398.5%	465.5%	542.3%	630.3%	731.2%	846.8%	979.3%	1131.1%	1305.0%
15	8.4%	32.2%	50.9%	69.7%	90.0%	112.5%	137.8%	166.6%	199.3%	236.7%	279.4%	328.3%	384.2%	448.1%	521.4%	605.2%	701.1%	810.8%	936.3%	1079.9%	1244.0%
20	8.3%	31.8%	50.3%	68.8%	88.7%	110.9%	135.8%	164.0%	196.1%	232.7%	274.6%	322.4%	377.0%	439.5%	511.0%	592.8%	686.2%	793.1%	915.2%	1054.8%	1214.2%
25	8.2%	31.6%	49.9%	68.2%	88.0%	109.9%	134.5%	162.5%	194.2%	230.4%	271.7%	318.9%	372.8%	434.4%	504.8%	585.4%	677.4%	782.5%	902.7%	1039.9%	1196.6%
30	8.2%	31.4%	49.6%	67.8%	87.5%	109.3%	133.7%	161.4%	192.9%	228.8%	269.7%	316.5%	369.9%	431.0%	500.8%	580.5%	671.5%	775.6%	894.4%	1030.0%	1184.9%
35	8.2%	31.3%	49.4%	67.6%	87.1%	108.8%	133.1%	160.7%	192.0%	227.7%	268.4%	314.8%	367.9%	428.6%	497.8%	577.0%	667.4%	770.6%	888.4%	1023.0%	1176.6%
40	8.2%	31.2%	49.3%	67.4%	86.8%	108.4%	132.7%	160.1%	191.3%	226.8%	267.3%	313.6%	366.4%	426.7%	495.6%	574.4%	664.2%	766.9%	884.0%	1017.8%	1170.4%
50	8.1%	31.1%	49.1%	67.1%	86.5%	107.9%	132.1%	159.3%	190.3%	225.6%	265.9%	311.8%	364.3%	424.2%	492.6%	570.7%	659.9%	761.7%	877.9%	1010.4%	1161.7%
75	8.1%	30.9%	48.8%	66.7%	86.0%	107.3%	131.2%	158.3%	189.1%	224.1%	264.0%	309.5%	361.5%	420.8%	488.5%	565.8%	654.1%	754.8%	869.7%	1000.7%	1150.2%
100	8.1%	30.9%	48.7%	66.5%	85.7%	107.0%	130.8%	157.8%	188.4%	223.3%	263.0%	308.3%	360.1%	419.1%	486.5%	563.4%	651.2%	751.3%	865.6%	995.9%	1144.5%
200	8.0%	30.7%	48.5%	66.2%	85.3%	106.5%	130.2%	157.0%	187.4%	222.1%	261.6%	306.6%	358.0%	416.6%	483.5%	559.8%	646.9%	746.2%	859.5%	988.7%	1135.9%
300	8.0%	30.7%	48.4%	66.1%	85.2%	106.3%	130.0%	156.7%	187.1%	221.7%	261.1%	306.0%	357.2%	415.7%	482.4%	558.6%	645.4%	744.5%	857.4%	986.2%	1133.1%

Table 2A: Buy and Hold Portfolio Return Skewness by Investment Horizon and Number of Stocks

Number Stocks	Investment Horizon (Months)																				
	1	12	24	36	48	60	72	84	96	108	120	132	144	156	168	180	192	204	216	228	240
5	0.00	0.99	1.59	2.18	2.80	3.52	4.37	5.39	6.66	8.24	10.24	12.79	16.07	20.31	25.83	33.03	42.46	54.86	71.16	92.67	121.05
10	0.00	0.90	1.43	1.93	2.44	3.01	3.67	4.44	5.37	6.51	7.93	9.71	11.98	14.88	18.64	23.52	29.89	38.25	49.24	63.74	82.89
15	0.00	0.87	1.37	1.83	2.29	2.80	3.37	4.04	4.82	5.77	6.93	8.37	10.19	12.50	15.47	19.31	24.30	30.84	39.43	50.76	65.74
20	0.00	0.85	1.33	1.77	2.21	2.68	3.21	3.81	4.51	5.35	6.36	7.61	9.16	11.13	13.63	16.86	21.04	26.50	33.66	43.09	55.56
25	0.00	0.83	1.31	1.73	2.16	2.61	3.10	3.66	4.31	5.08	5.99	7.11	8.50	10.23	12.43	15.25	18.88	23.62	29.81	37.96	48.73
30	0.00	0.83	1.29	1.71	2.12	2.56	3.03	3.56	4.17	4.89	5.74	6.76	8.03	9.60	11.58	14.10	17.34	21.55	27.04	34.26	43.79
35	0.00	0.82	1.28	1.69	2.10	2.52	2.98	3.49	4.07	4.75	5.55	6.51	7.68	9.12	10.94	13.24	16.19	20.00	24.96	31.46	40.04
40	0.00	0.82	1.27	1.68	2.08	2.49	2.94	3.43	3.99	4.64	5.40	6.31	7.41	8.76	10.45	12.57	15.29	18.78	23.32	29.26	37.09
50	0.00	0.81	1.26	1.66	2.05	2.45	2.88	3.35	3.88	4.48	5.19	6.02	7.02	8.23	9.73	11.60	13.97	17.01	20.93	26.03	32.73
75	0.00	0.80	1.24	1.63	2.01	2.39	2.80	3.24	3.72	4.27	4.90	5.62	6.47	7.49	8.72	10.23	12.11	14.48	17.50	21.40	26.46
100	0.00	0.80	1.24	1.62	1.99	2.36	2.76	3.18	3.64	4.16	4.74	5.41	6.19	7.10	8.19	9.51	11.13	13.15	15.68	18.92	23.09
200	0.00	0.79	1.22	1.60	1.95	2.32	2.69	3.09	3.52	3.99	4.51	5.09	5.75	6.50	7.36	8.38	9.58	11.03	12.80	14.98	17.70
300	0.00	0.79	1.22	1.59	1.94	2.30	2.67	3.06	3.48	3.93	4.43	4.98	5.59	6.29	7.08	7.99	9.05	10.30	11.79	13.60	15.81

Table 2B: Portfolio Return Skewness with Monthly Rebalancing, by Investment Horizon and Number of Stocks

Number Stocks	Investment Horizon (Months)																				
	1	12	24	36	48	60	72	84	96	108	120	132	144	156	168	180	192	204	216	228	240
5	0	0.90	1.42	1.88	2.33	2.81	3.31	3.86	4.46	5.13	5.88	6.72	7.67	8.74	9.96	11.35	12.93	14.73	16.80	19.17	21.88
10	0	0.84	1.32	1.73	2.13	2.54	2.96	3.42	3.91	4.44	5.02	5.67	6.38	7.17	8.05	9.04	10.14	11.38	12.77	14.33	16.09
15	0	0.82	1.28	1.68	2.06	2.45	2.85	3.27	3.73	4.22	4.76	5.35	6.00	6.71	7.50	8.38	9.35	10.44	11.65	13.01	14.52
20	0	0.81	1.26	1.65	2.03	2.40	2.79	3.20	3.64	4.12	4.63	5.19	5.81	6.49	7.24	8.06	8.98	10.00	11.13	12.39	13.80
25	0	0.81	1.25	1.64	2.00	2.38	2.76	3.16	3.59	4.06	4.56	5.10	5.70	6.36	7.08	7.88	8.76	9.74	10.83	12.04	13.38
30	0	0.80	1.25	1.63	1.99	2.36	2.74	3.14	3.56	4.02	4.51	5.04	5.63	6.27	6.98	7.76	8.62	9.58	10.63	11.81	13.11
35	0	0.80	1.24	1.62	1.98	2.35	2.72	3.12	3.54	3.99	4.47	5.00	5.58	6.21	6.91	7.68	8.52	9.46	10.49	11.65	12.93
40	0	0.80	1.24	1.61	1.97	2.34	2.71	3.10	3.52	3.96	4.45	4.97	5.54	6.17	6.86	7.61	8.45	9.37	10.39	11.53	12.78
50	0	0.79	1.23	1.61	1.96	2.32	2.69	3.08	3.49	3.93	4.41	4.93	5.49	6.11	6.78	7.53	8.34	9.25	10.25	11.36	12.59
75	0	0.79	1.22	1.60	1.95	2.31	2.67	3.05	3.46	3.89	4.36	4.87	5.42	6.02	6.68	7.41	8.21	9.09	10.06	11.14	12.34
100	0	0.79	1.22	1.59	1.94	2.30	2.66	3.04	3.44	3.87	4.34	4.84	5.39	5.98	6.64	7.35	8.14	9.01	9.97	11.04	12.21
200	0	0.78	1.21	1.58	1.93	2.28	2.64	3.02	3.42	3.84	4.30	4.80	5.34	5.92	6.56	7.27	8.04	8.90	9.84	10.88	12.03
300	0	0.78	1.21	1.58	1.93	2.28	2.64	3.01	3.41	3.83	4.29	4.78	5.32	5.90	6.54	7.24	8.01	8.86	9.79	10.82	11.97

Table 3: Portfolio Risk and Return over 240 Months

Panel A: Porfolio Return Standard Deviation

Number Stocks	Rebalance Frequency (Months)						
	1	12	24	48	60	120	240
5	1500%	1518%	1538%	1583%	1607%	1744%	2080%
10	1305%	1314%	1325%	1349%	1362%	1441%	1673%
15	1244%	1250%	1258%	1274%	1283%	1338%	1513%
20	1214%	1219%	1224%	1237%	1243%	1286%	1427%
25	1197%	1200%	1205%	1215%	1220%	1254%	1372%
30	1185%	1188%	1192%	1200%	1204%	1233%	1334%
35	1177%	1179%	1182%	1189%	1193%	1218%	1307%
40	1170%	1173%	1175%	1182%	1185%	1207%	1286%
50	1162%	1164%	1166%	1171%	1174%	1191%	1256%
75	1150%	1151%	1153%	1156%	1158%	1170%	1214%
100	1144%	1145%	1147%	1149%	1150%	1159%	1193%
200	1136%	1136%	1137%	1138%	1139%	1143%	1161%
300	1133%	1133%	1134%	1135%	1135%	1138%	1150%

Panel B: Porfolio Return Sharpe Ratio

Number Stocks	Rebalance Frequency (Months)						
	1	12	24	48	60	120	240
5	0.33	0.33	0.33	0.32	0.31	0.29	0.24
10	0.38	0.38	0.38	0.37	0.37	0.35	0.30
15	0.40	0.40	0.40	0.39	0.39	0.37	0.33
20	0.41	0.41	0.41	0.41	0.40	0.39	0.35
25	0.42	0.42	0.42	0.41	0.41	0.40	0.37
30	0.42	0.42	0.42	0.42	0.42	0.41	0.38
35	0.43	0.42	0.42	0.42	0.42	0.41	0.38
40	0.43	0.43	0.43	0.42	0.42	0.41	0.39
50	0.43	0.43	0.43	0.43	0.43	0.42	0.40
75	0.44	0.44	0.43	0.43	0.43	0.43	0.41
100	0.44	0.44	0.44	0.44	0.44	0.43	0.42
200	0.44	0.44	0.44	0.44	0.44	0.44	0.43
300	0.44	0.44	0.44	0.44	0.44	0.44	0.44

Table 3: Portfolio Risk and Return over 240 Months (Cont.)

Panel C: Portfolio Return Skewness Coefficient

Number Stocks	Rebalance Frequency (Months)						
	1	12	24	48	60	120	240
5	21.88	23.46	25.47	30.61	33.82	57.03	121.05
10	16.09	16.75	17.61	19.85	21.30	33.16	82.89
15	14.52	14.94	15.49	16.91	17.83	25.68	65.74
20	13.80	14.11	14.50	15.54	16.22	22.06	55.56
25	13.38	13.63	13.94	14.75	15.29	19.93	48.73
30	13.11	13.31	13.57	14.24	14.68	18.53	43.79
35	12.93	13.09	13.31	13.88	14.26	17.55	40.04
40	12.78	12.93	13.12	13.62	13.94	16.81	37.09
50	12.59	12.71	12.86	13.25	13.51	15.80	32.73
75	12.34	12.41	12.51	12.77	12.94	14.46	26.46
100	12.21	12.27	12.34	12.54	12.66	13.79	23.09
200	12.03	12.05	12.09	12.19	12.25	12.81	17.70
300	11.97	11.98	12.01	12.07	12.11	12.49	15.81

Panel D: Adjusted-for-Skewness Sharpe Ratio

Number Stocks	Rebalance Frequency (Months)						
	1	12	24	48	60	120	240
5	1.15	1.18	1.23	1.34	1.41	1.86	2.58
10	1.17	1.19	1.22	1.28	1.33	1.68	2.78
15	1.19	1.20	1.22	1.26	1.30	1.57	2.73
20	1.20	1.21	1.22	1.26	1.28	1.51	2.63
25	1.20	1.21	1.22	1.25	1.27	1.46	2.53
30	1.20	1.21	1.22	1.24	1.26	1.43	2.43
35	1.21	1.21	1.22	1.24	1.26	1.40	2.34
40	1.21	1.21	1.22	1.24	1.25	1.38	2.27
50	1.21	1.22	1.22	1.24	1.25	1.35	2.14
75	1.22	1.22	1.22	1.23	1.24	1.31	1.91
100	1.22	1.22	1.22	1.23	1.24	1.29	1.78
200	1.22	1.22	1.22	1.23	1.23	1.26	1.53
300	1.22	1.22	1.22	1.23	1.23	1.25	1.44

Table 4: Portfolio Risk and Return over 120 Months

Panel A: Porfolio Return Standard Deviation

Number Stocks	Rebalance Frequency (Months)				
	1	12	24	60	120
5	319%	321%	324%	334%	353%
10	289%	291%	292%	298%	310%
15	279%	280%	282%	286%	294%
20	275%	275%	276%	279%	286%
25	272%	272%	273%	276%	281%
30	270%	270%	271%	273%	278%
35	268%	269%	269%	271%	275%
40	267%	268%	268%	270%	273%
50	266%	266%	267%	268%	271%
75	264%	264%	264%	265%	267%
100	263%	263%	263%	264%	266%
200	262%	262%	262%	262%	263%
300	261%	261%	261%	261%	262%

Panel B: Porfolio Return Sharpe Ratio

Number Stocks	Rebalance Frequency (Months)				
	1	12	24	48	120
5	0.46	0.45	0.45	0.43	0.41
10	0.50	0.50	0.50	0.49	0.47
15	0.52	0.52	0.52	0.51	0.49
20	0.53	0.53	0.53	0.52	0.51
25	0.53	0.53	0.53	0.53	0.52
30	0.54	0.54	0.54	0.53	0.52
35	0.54	0.54	0.54	0.54	0.53
40	0.54	0.54	0.54	0.54	0.53
50	0.55	0.55	0.54	0.54	0.54
75	0.55	0.55	0.55	0.55	0.54
100	0.55	0.55	0.55	0.55	0.55
200	0.55	0.55	0.55	0.55	0.55
300	0.56	0.56	0.56	0.56	0.55

Table 4: Portfolio Risk and Return over 120 Months (Cont.)

Panel C: Portfolio Return Skewness Coefficient

Number Stocks	Rebalance Frequency (Months)				
	1	12	24	60	120
5	5.88	6.14	6.45	7.64	10.24
10	5.02	5.16	5.33	6.03	7.93
15	4.76	4.85	4.97	5.47	6.93
20	4.63	4.70	4.79	5.18	6.36
25	4.56	4.61	4.69	5.00	5.99
30	4.51	4.56	4.62	4.88	5.74
35	4.47	4.51	4.57	4.79	5.55
40	4.45	4.48	4.53	4.73	5.40
50	4.41	4.44	4.48	4.64	5.19
75	4.36	4.38	4.41	4.52	4.90
100	4.34	4.35	4.37	4.45	4.74
200	4.30	4.31	4.32	4.36	4.51
300	4.29	4.29	4.30	4.33	4.43

Panel D: Adjusted-For-Skewness Sharpe Ratio

Number Stocks	Rebalance Frequency (Months)				
	1	12	24	60	120
5	0.86	0.87	0.88	0.92	0.99
10	0.92	0.93	0.94	0.96	1.05
15	0.95	0.95	0.96	0.98	1.06
20	0.96	0.96	0.97	0.99	1.05
25	0.97	0.97	0.97	0.99	1.05
30	0.97	0.97	0.98	0.99	1.04
35	0.98	0.98	0.98	0.99	1.04
40	0.98	0.98	0.98	0.99	1.04
50	0.98	0.98	0.99	1.00	1.03
75	0.99	0.99	0.99	1.00	1.02
100	0.99	0.99	0.99	1.00	1.02
200	1.00	1.00	1.00	1.00	1.01
300	1.00	1.00	1.00	1.00	1.01

Figure 1A: Portfolio Return Standard Deviation, Buy and Hold

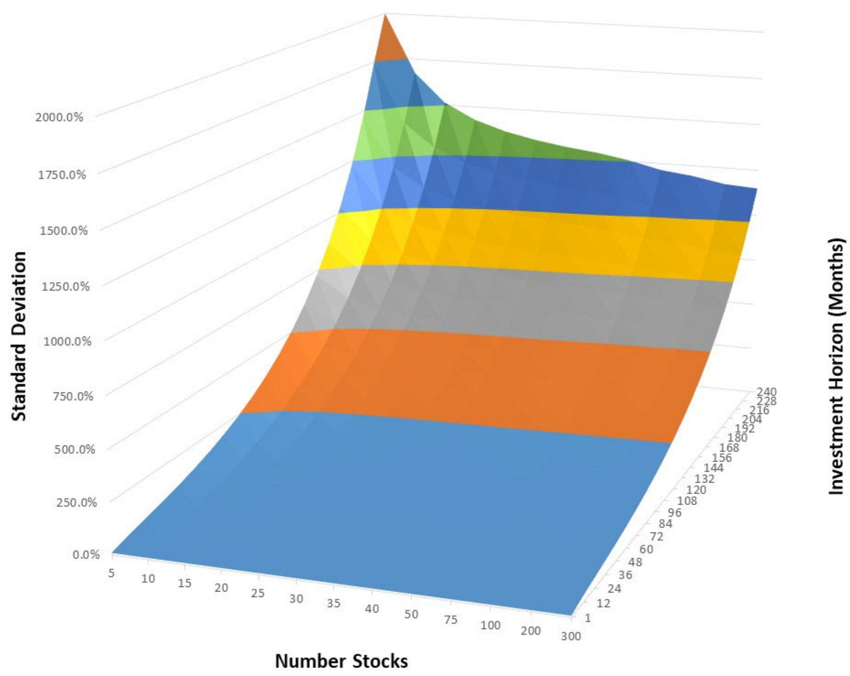


Figure 1B: Portfolio Return Standard Deviation, with Monthly Rebalancing

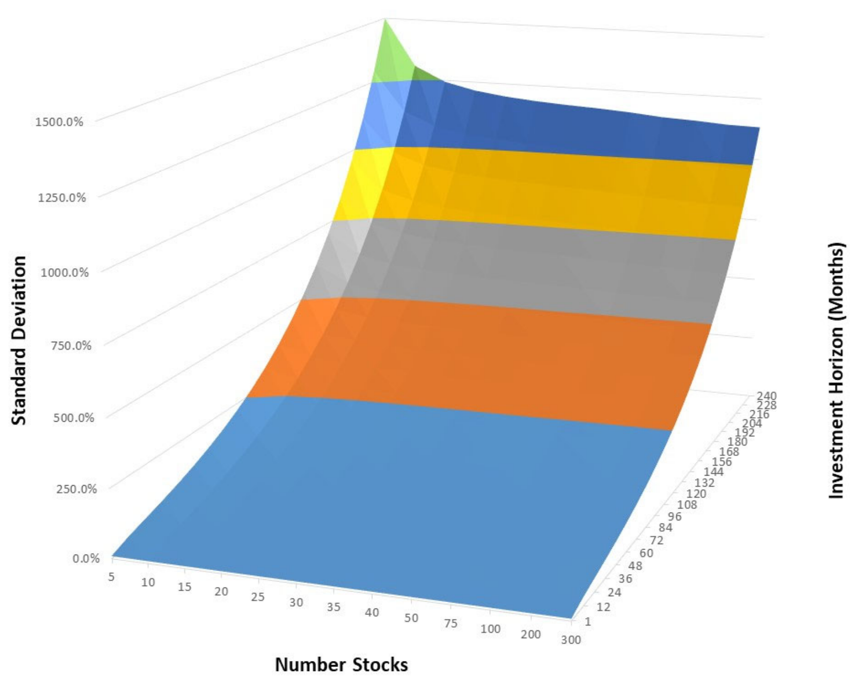


Figure 2A: Portfolio Return Skewness Coefficient, Buy-and-Hold

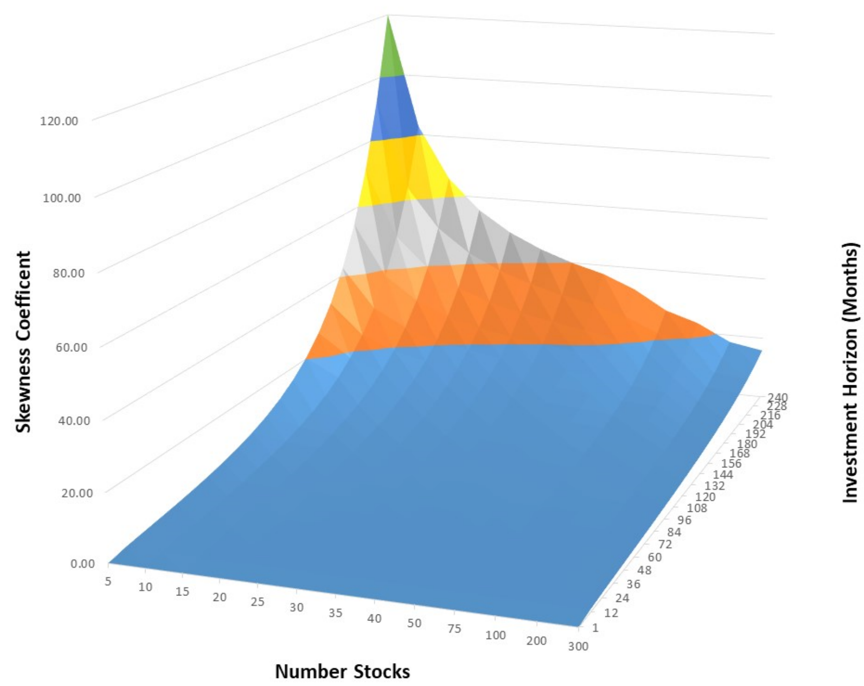
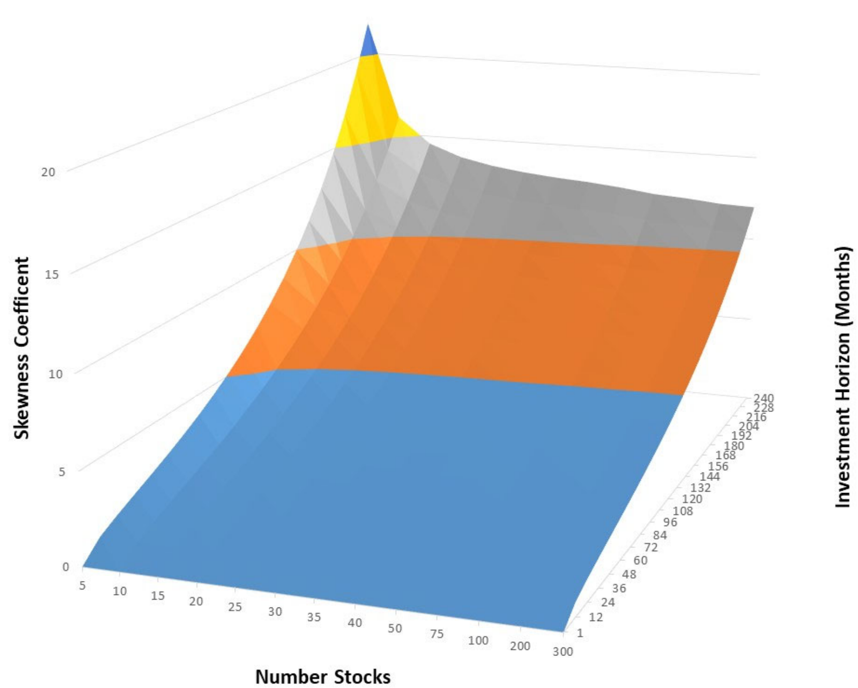


Figure 2B: Portfolio Return Skewness Coefficient, Monthly Rebalancing



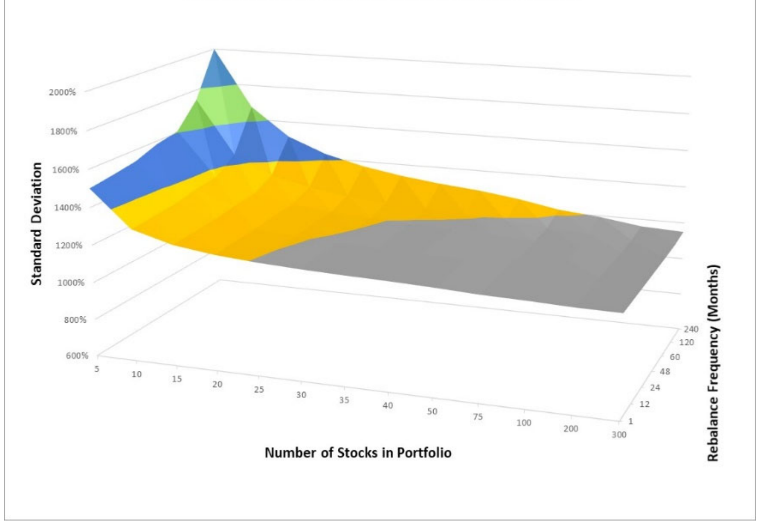


Figure 3A: Standard Deviation of 240 Month Portfolio Returns

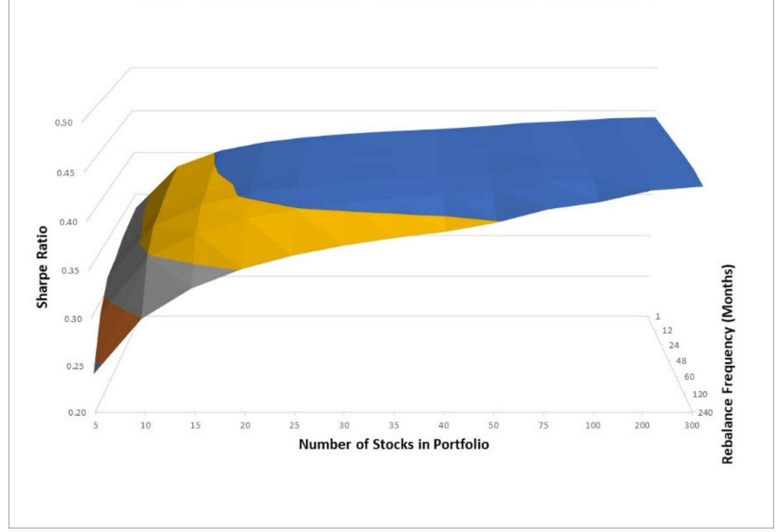


Figure 3B: Sharpe Ratios of 240 Month Portfolio Returns (Mean Return = 500.91%)

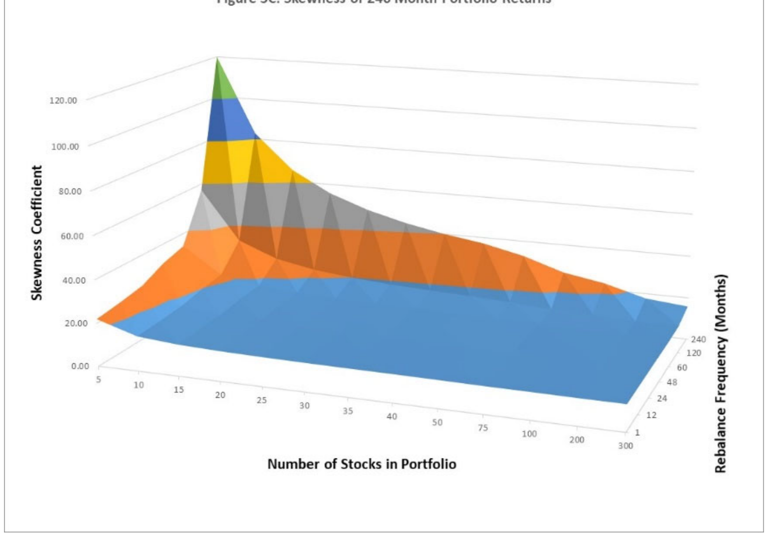


Figure 3C: Skewness of 240 Month Portfolio Returns

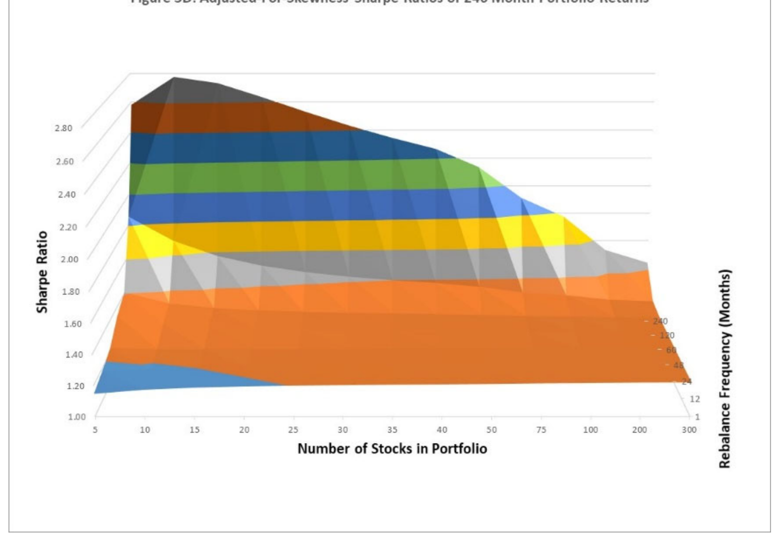


Figure 3D: Adjusted-For-Skewness Sharpe Ratios of 240 Month Portfolio Returns

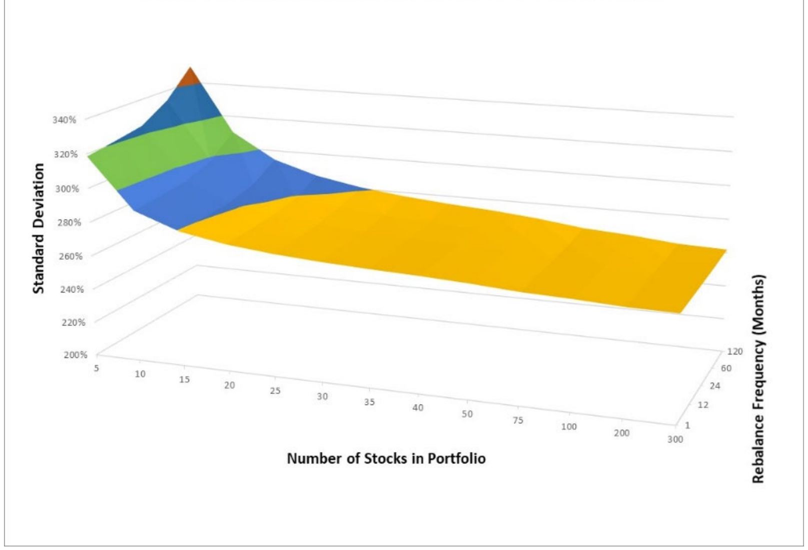


Figure 4A: Standard Deviation of 120 Month Portfolio Returns

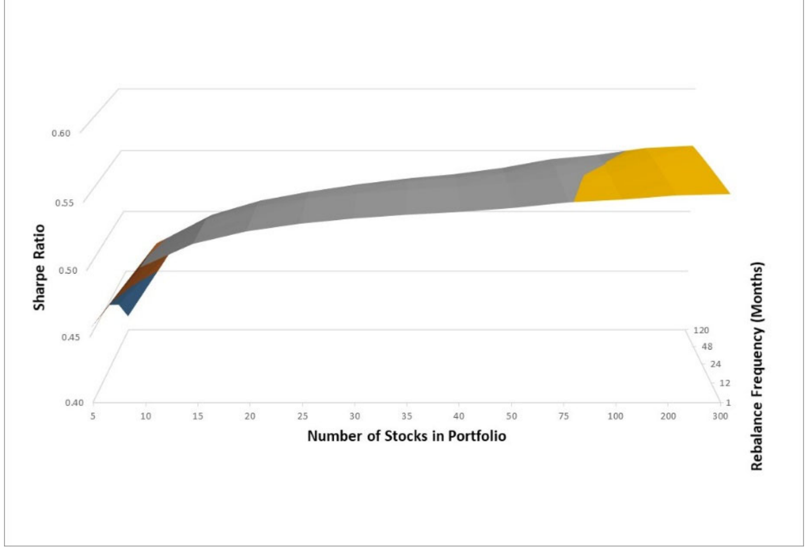


Figure 4B: Sharpe Ratios of 120 Month Portfolio Returns (Mean Return = 145.14%)

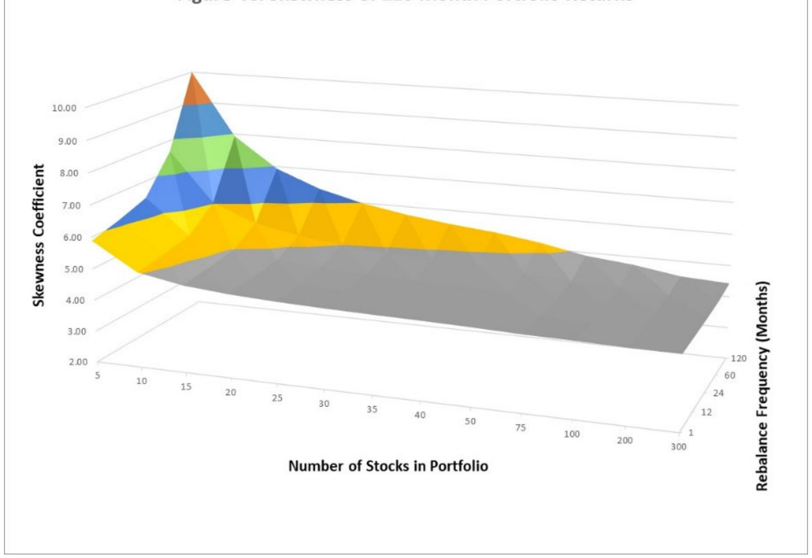


Figure 4C: Skewness of 120 Month Portfolio Returns

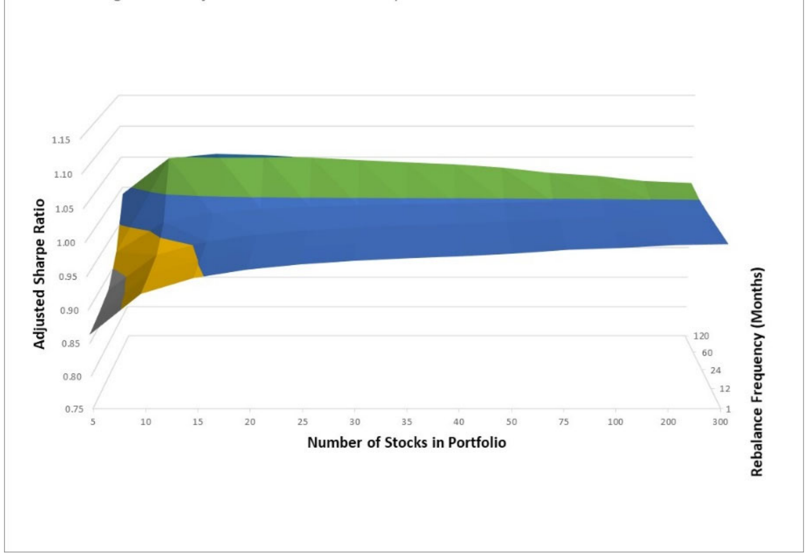


Figure 4D: Adjusted-for-Skewness Sharpe Ratios of 120 Month Portfolio Returns